NORTH SYDNEY GIRLS HIGH SCHOOL

2020 | **HSC Trial Examination**

Mathematics Extension 1

General Instructions

- Reading Time – 10 minutes
- Working Time – 2 hours
- Write using black pen
- NESA approved calculators may be used
- A reference sheet is provided
- In Questions 11 – 14, show relevant mathematical reasoning and/or calculations

Total marks – 70

Section I – 10 marks (pages 2 – 5)

- Attempt Questions 1 – 10
- Allow about 15 minutes for this section

Section II – 60 marks (pages 6 – 11)

- Attempt Questions 11 – 14
- Allow about 1 hours 45 minutes for this section

NAME:______________________________ **TEACHER:**____________________________

STUDENT NUMBER:

Question	1-10	11	12	13	14	Total
Mark	/10	/15	/15	/15	/15	/70

Section I

10 marks
Attempt Questions 1–10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1–10.

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1 What is the value of $\cos^{-1}\left(-\frac{1}{2}\right)$?

A. $\frac{\pi}{3}$

B. $-\frac{\pi}{3}$

C. $\frac{2\pi}{3}$

D. $\frac{4\pi}{3}$

2 Which integral is obtained when the substitution $u = 1 + 3x$ is applied to $\int x\sqrt{1+3x}\, dx$?

A. $\frac{1}{9}\int (u-1)\sqrt{u}\, du$

B. $\frac{1}{6}\int (u-1)\sqrt{u}\, du$

C. $\frac{1}{3}\int (u-1)\sqrt{u}\, du$

D. $\frac{1}{4}\int (u-1)\sqrt{u}\, du$

3 Consider the vectors $\underset{\sim}{u} = 2\underset{\sim}{i} - 5\underset{\sim}{j}$, $\underset{\sim}{v} = \underset{\sim}{i} + a\underset{\sim}{j}$ and $\underset{\sim}{w} = \underset{\sim}{i} + b\underset{\sim}{j}$.

If $\underset{\sim}{u} \parallel \underset{\sim}{v}$ and $\underset{\sim}{u} \perp \underset{\sim}{w}$, what are possible values for a and b?

A. $a = \frac{5}{2},\ b = \frac{2}{5}$

B. $a = \frac{5}{2},\ b = -\frac{2}{5}$

C. $a = -\frac{5}{2},\ b = \frac{2}{5}$

D. $a = -\frac{5}{2},\ b = -\frac{2}{5}$

4. Using the trigonometric product-to-sum results, what is $\int \sin 3x \sin x\, dx$?

A. $\frac{1}{4}\sin 2x - \frac{1}{8}\sin 4x$

B. $\frac{1}{8}\sin 4x - \frac{1}{4}\sin 2x$

C. $\frac{1}{4}\cos 2x - \frac{1}{8}\cos 4x$

D. $\frac{1}{8}\cos 4x - \frac{1}{4}\cos 2x$

5 Consider the equation $x^2 + 2x + 3 = 0$.

What is the sum of the reciprocals of the roots of this equation?

A. $\frac{2}{3}$

B. $-\frac{2}{3}$

C. $\frac{1}{2}$

D. $-\frac{1}{2}$

6 From the set of integers $1, 2, 3, \ldots, 999$, numbers are selected one at a time without repetition. How many numbers need to be chosen to guarantee that we have chosen at least four multiples of 9?

A. 37

B. 444

C. 445

D. 892

7 If $f(x) = \dfrac{3+e^{2x}}{5}$, which of the following is an expression for $f^{-1}(x)$?

A. $\ln(5x-3)$

B. $\dfrac{1}{2}\ln(5x-3)$

C. $\ln 5x - \ln 3$

D. $\dfrac{1}{2}(\ln 5x - \ln 3)$

8 $x = 2$ is a zero of multiplicity $r > 1$ of the polynomial $P(x) = (x-2)Q(x)$, where $Q(x)$ is a polynomial.

Which of the following statements need not be true?

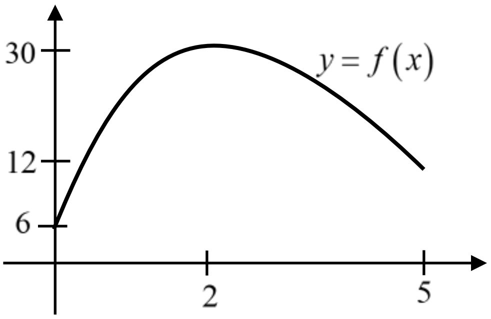
A. $P(2) = 0$

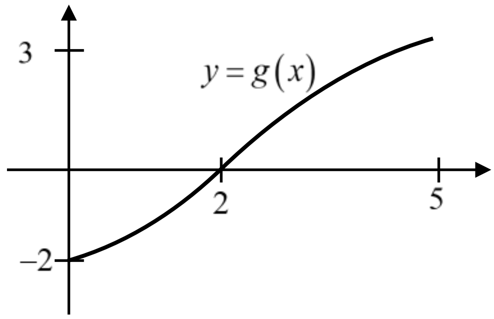
B. $Q(2) = 0$

C. $P'(2) = 0$

D. $Q'(2) = 0$

9 Consider the following sketches of the functions $y = f(x)$ and $y = g(x)$.





What is the range of the function $y = \dfrac{f(x)}{g(x)}$?

A. $(-\infty, -3] \cup [4, \infty)$

B. $[-3, 4]$

C. $[-15, 2]$

D. all real y

10 The continuous function $f(x+a)$ is even, where $a > 0$, and is defined for all real x.

Associated with the function $f(x)$ is an inverse function $f^{-1}(x)$, which has range $(-\infty, a]$.

If $c > a$, what is the value of $f^{-1}\left[f(c)\right]$?

A. $2a - c$

B. $a - c$

C. $-c$

D. $-a - c$

End of Section I

Section II

Total marks − 60
Attempt Questions 11−14
Allow about 1 hour 45 minutes for this section.

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

In Questions 11 to 14, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use a SEPARATE writing booklet.

(a) Find $\int \frac{dx}{x^2+9}$. **1**

(b) Find a general solution to $\sin x = \frac{\sqrt{3}}{2}$. **2**

(c) Solve for x: $\frac{1-2x}{1+x} \geq 1$. **3**

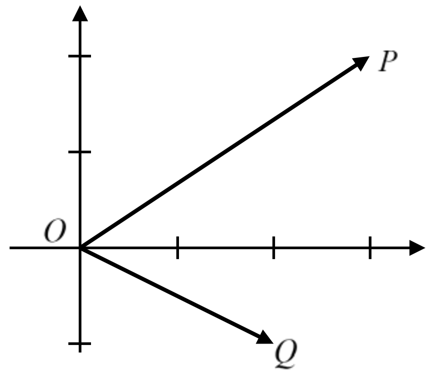
(d) Donald, Melania and five other people get on a bus one at a time. **2**

In how many ways can the seven people board the bus if Melania refuses to get on immediately following Donald?

(e) Find the term independent of x in the expansion of $\left(2x^2 - \frac{3}{x}\right)^9$. **2**

Question 11 continues on page 7

(f) Consider the vectors $\underset{\sim}{u} = \overrightarrow{OP} = 3\underset{\sim}{i} + 2\underset{\sim}{j}$ and $\underset{\sim}{v} = \overrightarrow{OQ} = 2\underset{\sim}{i} - \underset{\sim}{j}$.



(i) Find the vector projection $\overrightarrow{OR}$ of $\underset{\sim}{v}$ onto $\underset{\sim}{u}$. **2**

(ii) Use vector methods to find the exact area of $\triangle OPQ$. **3**

End of Question 11

Question 12 (15 marks) Use a SEPARATE writing booklet.

(a) When the polynomial $P(x)$ is divided by $x^2 - 16$ the remainder is $3x - 1$. **2**

What is the remainder when $P(x)$ is divided by $x - 4$?

(b) Use mathematical induction to prove that $4^n + 14$ is divisible by 6 for all positive integers n. **3**

(c) An exam sat by 130 students was marked out of 25 marks. No half marks were awarded and exactly four students scored full marks. **2**

A total of N students received the modal mark.

What is the smallest value of N ? Justify your answer.

(d) (i) Find $\dfrac{d}{dx}\left[\dfrac{2x}{4+x^2} + \tan^{-1}\dfrac{x}{2}\right]$, writing your answer in the form $\dfrac{A}{\left(4+x^2\right)^2}$, **3**

where A is a constant.

(ii) Hence evaluate $\displaystyle\int_0^2 \frac{dx}{\left(4+x^2\right)^2}$. **2**

(e) (i) Sketch the curve $y = \sin^{-1}\dfrac{4x-1}{2}$. **2**

(ii) Hence or otherwise solve $\sin^{-1}\dfrac{4|x|-1}{2} > 0$. **1**

End of Question 12

Question 13 (15 marks) Use a SEPARATE writing booklet.

(a) Find the exact value of $\tan\left(2\cos^{-1}\dfrac{12}{13}\right)$. **2**

(b) Use the substitution $u=\sqrt{1+x}$ to find $\displaystyle\int \frac{5x^2+10x}{\sqrt{1+x}}\,dx$. **3**

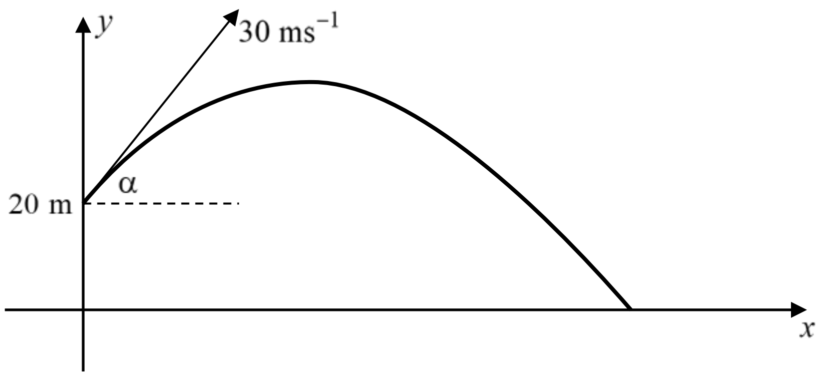
(c) A 4-digit number begins with 1 and has exactly one pair of identical digits (not necessarily adjacent).

(i) How many such numbers can be formed if the repeated digit is 1? **1**

(ii) How many such numbers can be formed in total? **2**

(d) A projectile is fired from a point 20 metres above level ground, with a velocity of 30 ms^{-1} and at an angle of $\alpha°$ above the horizontal.

The displacement, velocity and acceleration of the projectile at any time are given by the displacement vector $\underset{\sim}{r}$, the velocity vector $\underset{\sim}{v}$ and the acceleration vector $\underset{\sim}{a}$ respectively. The initial velocity of the projectile is represented by the vector $\underset{\sim}{u}$.



The acceleration vector is $\underset{\sim}{a}=-10\underset{\sim}{j}$ and has units of ms^{-2}.

(i) Write down, in component form, the initial velocity vector $\underset{\sim}{u}$. **1**

(ii) Derive the displacement vector $\underset{\sim}{r}$ in component form. **3**

(iii) By showing that the Cartesian equation of the trajectory is $y=x\tan\alpha-\dfrac{x^2}{180}\sec^2\alpha+20$, or otherwise, find the value(s) of α for which the projectile will hit a target 90 metres down-range and 1 metre above the ground. **3**

End of Question 13

Question 14 (15 marks) Use a SEPARATE writing booklet.

(a) By writing $4\sin\theta + 5\cos\theta$ in the form $R\sin(\theta+\alpha)$, or otherwise, find all solutions to $4\sin\theta + 5\cos\theta = 3$ in the interval $[0, 2\pi]$. **3**

(b) The rate at which water evaporates from a pool is proportional to the volume of water remaining in the pool, that is $\dfrac{dV}{dt} = k(V_0 - V)$, where V is the volume of water in litres which has evaporated, V_0 is the initial volume of water in the pool, k is a constant, and t is the time in days since evaporation began.

(i) Show that $V = V_0\left(1 - e^{-kt}\right)$ satisfies the above differential equation. **1**

(ii) It takes 5 days for one quarter of the original volume of water to evaporate. **3**

Determine how much longer it will take for the second quarter of the original volume to evaporate.

Give your answer in days, correct to one decimal place.

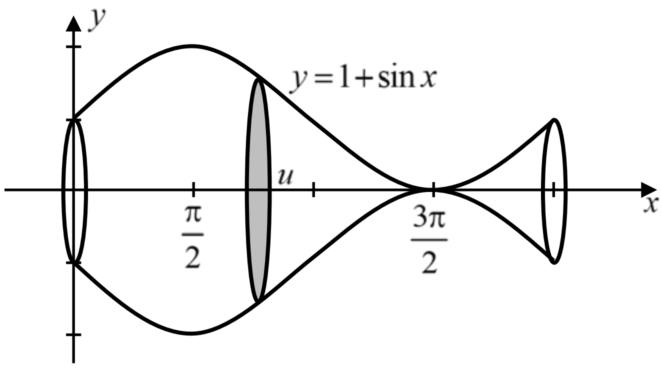
(c) By considering a group of students consisting of n boys and n girls, or otherwise, prove the identity: **3**

$$\binom{2n}{3} = 2\binom{n}{3} + 2n\binom{n}{2}.$$

Question 14 continues on page 11

(d) The diagram shows the graph of $y = 1 + \sin x$ in the domain $[0, 2\pi]$, which has been rotated about the x-axis to form a wine glass. All measurements are in centimetres.

The part of the glass in the interval $\left[0, \frac{3\pi}{2}\right]$ can contain wine. The part in the interval $\left[\frac{3\pi}{2}, 2\pi\right]$ is only the support, and cannot hold wine.



After sitting the glass on a table upright on its support, wine is poured into the glass up to the level $x = u$.

(i) Show that the volume of wine in the glass is given by: **3**

$$V = \frac{\pi}{4}\left(9\pi - 6u + 8\cos u + \sin 2u\right).$$

(ii) If wine is poured into the glass at a constant rate of 0.5 cm/s, at what rate will the depth of wine be increasing when the depth is $\frac{\pi}{2}$ cm? **2**

End of paper

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2020 Extension 1 Trial Solutions

Multiple Choice

Summary: 1. C 2. A 3. C 4. A 5. B
6. D 7. B 8. D 9. A 10. A

1. $\cos^{-1}\left(-\frac{1}{2}\right) = \pi - \cos^{-1}\frac{1}{2}$

$= \pi - \frac{\pi}{3}$

$= \frac{2\pi}{3}$

2. $I = \int x\sqrt{1+3x}\,dx$

$= \int \frac{u-1}{3}\sqrt{u}\cdot\frac{1}{3}du$

$= \frac{1}{9}\int (u-1)\sqrt{u}\,du$

$u = 1+3x \Rightarrow x = \frac{u-1}{3}$

$du = 3dx \Rightarrow dx = \frac{1}{3}du$

3. $\underset{\sim}{u}\cdot\underset{\sim}{w} = 0$

$(2\underset{\sim}{i} - 5\underset{\sim}{j})(\underset{\sim}{i} + b\underset{\sim}{j}) = 0$

$2 - 5b = 0$

$b = \frac{2}{5}$

$\underset{\sim}{v} = \frac{1}{2}\underset{\sim}{a}$

$\therefore a = -\frac{5}{2}$

4. $\sin A\sin B = \frac{1}{2}\left[\cos(A-B) - \cos(A+B)\right]$

$\sin 3x\sin x = \frac{1}{2}\left[\cos(3x-x) - \cos(3x+x)\right]$

$\int \sin 3x\sin 5x = \frac{1}{2}\int(\cos 2x - \cos 4x)dx$

$= \frac{1}{2}\left(\frac{1}{2}\sin 2x - \frac{1}{4}\sin 4x\right) + c$

$= \frac{1}{4}\sin 2x - \frac{1}{8}\sin 4x + c$

5. $\frac{1}{\alpha} + \frac{1}{\beta} = \frac{\alpha+\beta}{\alpha\beta}$

$= \frac{-2}{3}$

6. 888 non-multiples + 4 multiples

$= 892$

⑦ $y = \dfrac{3+e^{2x}}{5}$

Inv: $x = \dfrac{3+e^{2y}}{5}$

$5x = 3 + e^{2y}$

$5x - 3 = e^{2y}$

$2y = \ln(5x-3)$

$y = \frac{1}{2}\ln(5x-3)$

⑧ A is true as 2 is a zero

B is true as 2 is a multiple zero

C is true as 2 is a multiple zero

D is true ONLY if 2 has multiplicity greater than 2

⑨ Case I: $x \in [0,2) \Rightarrow$ max y-value $= \frac{6}{-2} = -3$

as $x \to 2^-$, $y \to \frac{30}{0^-} = -\infty$

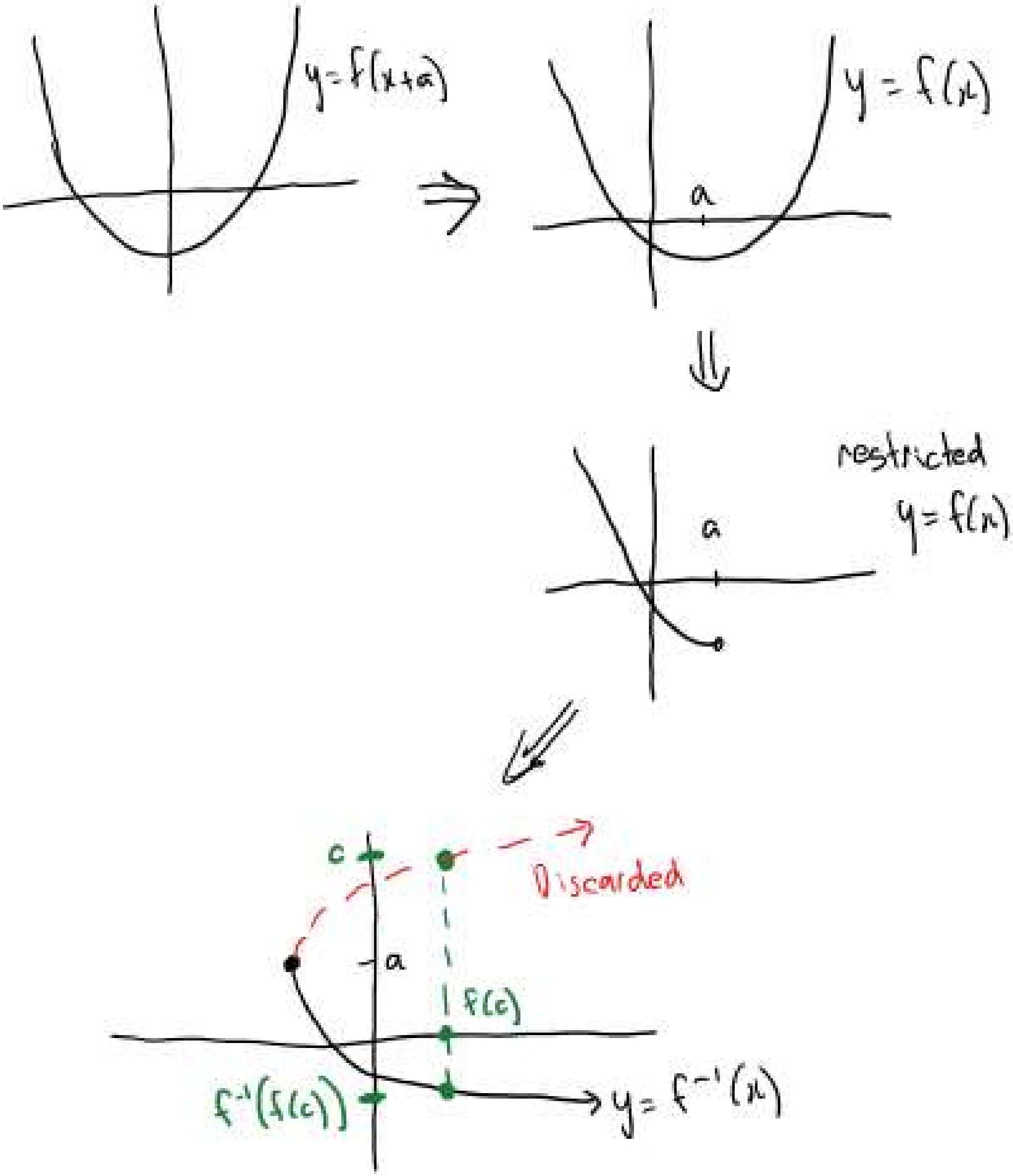
$\therefore y \in (-\infty, -3]$

Case II: $x \in (2,5] \Rightarrow$ min y-value $= \frac{12}{3} = 4$

as $x \to 2^+$, $y \to \frac{30}{0^+} = +\infty$

$\therefore y \in [4, \infty)$

(10)



By symmetry: $c - a = a - f^{-1}(f(c))$

$$f^{-1}(f(c)) = 2a - c$$

Question 11

(a) $\displaystyle\int \frac{dx}{x^2+9} = \boxed{\frac{1}{3}\tan^{-1}\frac{x}{3}}$ [1]

Generally well done

(b) $\sin x = \frac{\sqrt{3}}{2}$

$\boxed{x = \frac{\pi}{3} + 2k\pi, \quad \frac{2\pi}{3} + 2k\pi \quad (\text{where } k \text{ is an integer})}$ [2]

Generally well done. A few students did not know how to find the general solution.

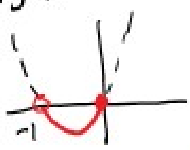
(c) $\dfrac{1-2x}{1+x} \geqslant 1$

$[\times (1+x)^2]$ $\quad (1-2x)(1+x) \geqslant (1+x)^2, \quad x \neq -1$

$(1+x)^2 - (1+x)(1-2x) \leqslant 0$

$(1+x)[(1+x) - (1-2x)] \leqslant 0$

$3x(1+x) \leqslant 0$



$\boxed{-1 < x \leqslant 0}$

OR: $\boxed{x \in (-1, 0]}$ [3]

Generally well done. Some students forgot $x \neq -1$.

(d) No restrictions: $7!$ orderings

Melania immediately behind Trumpler:

$\boxed{TM}$ _ _ _ _ _ $\quad 6!$ orderings

Answer: $7! - 6! = \boxed{4320}$ [2]

Generally well done.

(e) $\left(2x^2 - 3x^{-1}\right)^9$

$$T_k = \binom{9}{k}\left(2x^2\right)^{9-k}\left(-3x^{-1}\right)^k, \quad k = 0, 1, 2, \ldots, 9$$

$$= \binom{9}{k} \cdot 2^{9-k} \cdot x^{18-2k} \cdot (-3)^k \cdot x^{-k}$$

$$= \binom{9}{k} \cdot 2^{9-k} \cdot (-3)^k \cdot x^{18-3k}$$

Indept. of x: $18 - 3k = 0 \Rightarrow k = 6$

$$T_6 = \binom{9}{6} \cdot 2^3 \cdot (-3)^6 \cdot x^0 = \boxed{489\,888} \quad [2]$$

Generally well done.

(f) (i) $$\text{proj}_{\underset{\sim}{u}}(\underset{\sim}{v}) = \frac{\underset{\sim}{v} \cdot \underset{\sim}{u}}{\underset{\sim}{u} \cdot \underset{\sim}{u}}\underset{\sim}{u}$$

$$= \frac{(2)(3) + (-1)(2)}{(3)(3) + (2)(2)}(3\underset{\sim}{i} + 2\underset{\sim}{j})$$

$$= \frac{4}{13}(3\underset{\sim}{i} + 2\underset{\sim}{j})$$

$$\overrightarrow{OR} = \boxed{\frac{12}{13}\underset{\sim}{i} + \frac{8}{13}\underset{\sim}{j}} \quad [2]$$

Some students used the wrong formula to find the vector projection $\overrightarrow{OR}$ of $\underset{\sim}{v}$ onto $\underset{\sim}{u}$.

(ii) $$\overrightarrow{RQ} = \overrightarrow{OQ} - \overrightarrow{OR}$$

$$= (2\underset{\sim}{i} - \underset{\sim}{j}) - \left(\frac{12}{13}\underset{\sim}{i} + \frac{8}{13}\underset{\sim}{j}\right)$$

$$= \frac{14}{13}\underset{\sim}{i} - \frac{21}{13}\underset{\sim}{j}$$

$$= 7\left(\frac{2}{13}\underset{\sim}{i} - \frac{3}{13}\underset{\sim}{j}\right) \Rightarrow |\overrightarrow{RQ}| = 7 \cdot \sqrt{\left(\frac{2}{13}\right)^2 + \left(\frac{-3}{13}\right)^2}$$

$$= \frac{7}{\sqrt{13}}$$

$$|\overrightarrow{OP}| = \sqrt{2^2 + 3^2} = \sqrt{13}$$

$$\text{Area} = \frac{1}{2} \times \sqrt{13} \times \frac{7}{\sqrt{13}} = \boxed{\frac{7}{2}\text{ units}^2} \quad [3]$$

Many students did not know $\overrightarrow{QR} \perp \overrightarrow{OP}$.

Many students did not use : $\overrightarrow{RQ} = \overrightarrow{OQ} - \overrightarrow{OR}$

or $\overrightarrow{QR} = \overrightarrow{OR} - \overrightarrow{OQ}$

Question 12

(a) $P(x) = (x^2-16)Q(x) + (3x-1)$

remainder $= P(4)$

$= 0 \cdot Q(x) + 11$

$= \boxed{11}$

[2]

Students who got a remainder involving x should have realised that when dividing by a linear polynomial the remainder is at most a constant.

Many students could not correctly write the division transformation. Many students who wrote it correctly wasted time by dividing $x-4$ into the remainder instead of using the remainder theorem.

(b) RTP: $6 \mid 4^n + 14$, $n = 1, 2, 3, \ldots$

Test $n=1$: $4^1 + 14 = 18 = 6(3)$

$\therefore$ true for $n=1$

Assume true for $n=k$:

$4^k + 14 = 6M$ (M an integer)

Prove true for $n=k+1$:

$4^{k+1} + 14 = 4(4^k) + 14$

$= 4(6M-14) + 14$ (by assumption)

$= 24M - 42$

$= 6(4M-7)$ ($4M-7$ an integer)

$\therefore$ true for $n=k+1$ when true for $n=k$

$\therefore$ $6 \mid 4^n + 14$ for $n = 1, 2, 3, \ldots$ by mathematical induction.

[3]

Generally well done.

Make sure you specify that M (see solutions) is an integer.

Try to get the language right. "Assume $n=k$ is true" is not meaningful. It should be "Assume (the statement is) true for $n=k$".

(c) Remove the known results – 4 scores of 25

Categories: the 25 remaining possible marks – 0 to 24

Objects to be categoried: the 126 remaining students $= 5 \times 25 + 1$

As there are more objects than 5 times the number of categories, at least one category must contain 6 objects

ie. at least one mark was awarded to at least six students [2]

Many students believed this was a question on statistics and attempted to invoke the normal distribution. Learn to recognise a question on the pigeonhole principle.

A number of students correctly divided 126 by 25, rounded up, and then added another 1. The rounding up IS the PHP.

A majority of students did not correctly count the number of categories (pigeonholes) because they didn't allow for a mark of zero.

(d) (i)
$$\frac{d}{dx}\left[\frac{2x}{4+x^2} + \tan^{-1}\frac{x}{2}\right] = \frac{(4+x^2)\cdot 2 - 2x\cdot 2x}{(4+x^2)^2} + \frac{1}{1+\left(\frac{x}{2}\right)^2}\cdot\frac{1}{2}$$
$$= \frac{8-2x^2}{(4+x^2)^2} + \frac{2}{4+x^2}$$
$$= \frac{8-2x^2+2(4+x^2)}{(4+x^2)^2}$$
$$= \boxed{\frac{16}{(4+x^2)^2}}$$
[3]

Generally well done. A few students integrated the first term instead of differentiating, while many made careless algebraic errors.

(ii)
$$\int_0^2 \frac{dx}{(4+x^2)^2} = \frac{1}{16}\int_0^2 \frac{16}{(4+x^2)^2}\,dx$$
$$= \frac{1}{16}\left[\frac{2x}{4+x^2} + \tan^{-1}\frac{x}{2}\right]_0^2$$
$$= \frac{1}{16}\left(\frac{4}{4+4} + \frac{\pi}{4} - 0 - 0\right)$$
$$= \boxed{\frac{\pi}{64} + \frac{1}{32}}$$
[2]

If you weren't able to successfully complete part (i), full marks were still accessible for this question by using A instead of 16.

A few students had 16 outside the integral instead of $\frac{1}{16}$.

The value of $\tan^{-1}(1)$ is $\frac{\pi}{4}$, not 45. The inverse trig functions are always defined in radians.

(e) (i) $y = \sin^{-1} \frac{4x-1}{2}$

Domain: $-1 \leq \frac{4x-1}{2} \leq 1$

$-2 \leq 4x-1 \leq 2$

$-1 \leq 4x \leq 3$

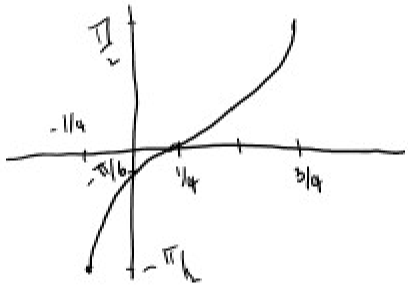
$-\frac{1}{4} \leq x \leq \frac{3}{4}$

Range: $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$

$(x=0)$: $y = \sin^{-1}\left(-\frac{1}{2}\right)$

$= -\sin^{-1}\left(\frac{1}{2}\right)$

$= -\frac{\pi}{6}$



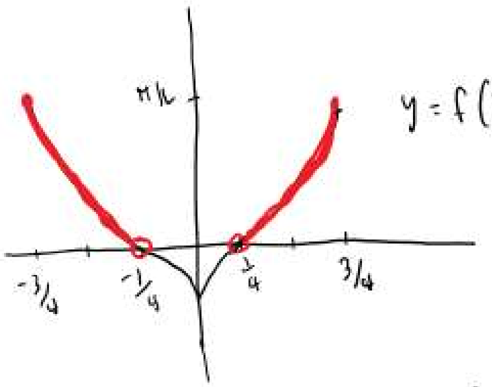
[2]

Be careful with the shape. A few curves looked nothing like an inverse sine curve.

Errors in the domain typically came from those who thought in terms of transformations instead of writing down and solving an inequality.

(ii) Let $f(x) = \sin^{-1} \frac{4x-1}{2}$

$\therefore f(|x|) = \sin^{-1} \frac{4|x|-1}{2}$



$f(|x|) > 0 \Rightarrow x \in \left[-\frac{3}{4}, \frac{1}{4}\right) \cup \left(\frac{1}{4}, \frac{3}{4}\right]$

[1]

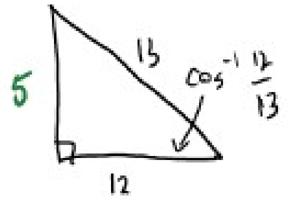
Many students did not consider the negative answers that arise from the absolute value.

Many did not consider the natural domain of the function displayed in your graph of part (i). Many of those who did, did not include the endpoints of the domain.

Most students belonged to at least one of those “manys”.

Question 13

(a) $\tan\left(2\cos^{-1}\frac{12}{13}\right) = \dfrac{2\tan\left(\cos^{-1}\frac{12}{13}\right)}{1-\tan^2\left(\cos^{-1}\frac{12}{13}\right)}$



$$= \frac{2\left(\frac{5}{12}\right)}{1-\left(\frac{5}{12}\right)^2} \times \frac{144}{144}$$

$$= \frac{120}{144-25}$$

$$= \boxed{\frac{120}{119}}$$

[2]

Was complete quite well. The errors that were made were:

- $\theta = 2\cos^{-1}\frac{12}{13}$ instead of $\theta = \cos^{-1}\frac{12}{13}$ and not being able to deal with the $\frac{\theta}{2}$
- Not considering that inverse cosine functions have an output of between zero and π. This resulted in having additional answers for the value of $\theta = \tan\left[\cos^{-1}\frac{12}{13}\right]$.

(b) $\displaystyle\int \frac{5x^2+10x}{\sqrt{1+x}}\,dx = \int \frac{5x(x+2)}{\sqrt{1+x}}\,dx$

$u = \sqrt{1+x}$

$u^2 = 1+x$

$x = u^2-1$

$dx = 2u\,du$

$$= \int \frac{5(u^2-1)(u^2+1)\cdot 2\cancel{u}\,du}{\cancel{u}}$$

$$= 10\int (u^4-1)\,du$$

$$= 10\left(\frac{u^5}{5} - u\right) + c$$

$$= 2u\left(u^4-5\right) + c$$

$$= 2\sqrt{1+x}\left[(1+x)^2-5\right] + c$$

$$= \boxed{2\left(x^2+2x-4\right)\sqrt{1+x} + c}$$

[3]

Quite well done for a tricky integration by substitution question, but lots of various algebraic errors that cropped up throughout. The main error was forgetting to write the resulting integral in terms of x instead of u.

(c) (i) Choose position for 2nd '1' : 3 ways

Choose 2 different ordered digits from remaining 9 to fill other 2 spots : ${}^9P_2$

$3 \times {}^9P_2 = \boxed{216}$ [1]

(ii) Case 2: '1' is not the repeated digit

Choose and place non-repeated digit : 9×3

Choose repeated digit to fill other 2 spots : 8

$9 \times 3 \times 8 = 216$

Total : $\boxed{432}$ [2]

Very poorly completed, very few students were able to obtain full marks. A significant proportion of the grade did not get any marks. Most of the errors indicated some major misconceptions with how the various counting techniques can be used.

There were quite a few minor errors also, such as:

- Not considering Zero as a possible digit to be used.
- Counting the rearranging of 3 objects, two of which were identical as 3!

(d) (i) $\boxed{\underset{\sim}{u} = 30\cos\alpha \, \underset{\sim}{i} + 30\sin\alpha \, \underset{\sim}{j}}$ [1]

The main error was that students developed the general velocity vector from from acceleration vector and left the initial velocity vector in terms of t, $\underset{\sim}{u} = (30\cos\alpha)\underset{\sim}{i} + (30\sin\alpha - 10t)\underset{\sim}{j}$ instead of just using the initial velocity of 30 m/s and direction of α above horizontal to produce $\underset{\sim}{u} = (30\cos\alpha)\underset{\sim}{i} + (30\sin\alpha)\underset{\sim}{j}$.

(ii) $\underset{\sim}{a} = -10\underset{\sim}{j}$

$\underset{\sim}{v} = -10t\,\underset{\sim}{j} + \underset{\sim}{c_1}$

$[t=0,\ \underset{\sim}{v} = 30\cos\alpha\,\underset{\sim}{i} + 30\sin\alpha\,\underset{\sim}{j}]$:

$\underset{\sim}{c_1} = 30\cos\alpha\,\underset{\sim}{i} + 30\sin\alpha\,\underset{\sim}{j}$

$\therefore \underset{\sim}{v} = 30\cos\alpha\,\underset{\sim}{i} + (30\sin\alpha - 10t)\underset{\sim}{j}$

$\underset{\sim}{r} = 30t\cos\alpha\,\underset{\sim}{i} + (30t\sin\alpha - 5t^2)\underset{\sim}{j} + c_2$

$[t=0,\ \underset{\sim}{r} = 20\underset{\sim}{j}]$:

$c_2 = 20\underset{\sim}{j}$

$\therefore \boxed{\underset{\sim}{r} = 30t\cos\alpha\,\underset{\sim}{i} + (30t\sin\alpha - 5t^2 + 20)\underset{\sim}{j}}$ [3]

Many students did not start from the acceleration vector and instead developed the displacement vector from $\underset{\sim}{v} = (30\cos\alpha)\underset{\sim}{i} + (30\sin\alpha - 10t)\underset{\sim}{j}$. Often students did not evaluate the constants of integration by substituting initial conditions.

(iii) $x = 30t\cos\alpha$ $\qquad$ $y = 30t\sin\alpha - 5t^2 + 20$

$t = \dfrac{x}{30\cos\alpha}$ $\qquad$ $= 30\left(\dfrac{x}{30\cos\alpha}\right)\sin\alpha - 5\left(\dfrac{x}{30\cos\alpha}\right)^2 + 20$

$y = x\tan\alpha - \dfrac{x^2}{180}\sec^2\alpha + 20$

$[x=90,\ y=1]$: $1 = 90\tan\alpha - 45(1+\tan^2\alpha) + 20$

$45\tan^2\alpha - 90\tan\alpha + 26 = 0$

$\tan\alpha = \dfrac{90 \pm \sqrt{90^2 - 4(45)(26)}}{90}$

$= 0.3502,\ 1.6498$

$\boxed{\alpha = 19°18',\ 58°47'}$ [3]

Quite well done across the grade.

Question 14

(a) Hence: Let $4\sin\theta + 5\cos\theta = R\sin(\theta + \alpha)$, $R > 0$

$$= R(\sin\theta\cos\alpha + \cos\theta\sin\alpha)$$

$$= (R\cos\alpha)\sin\theta + (R\sin\alpha)\cos\theta$$

Equating coefficients of $\sin\theta$ & $\cos\theta$:

$R\cos\alpha = 4$ ①

$R\sin\alpha = 5$ ②

①^2 + ②^2: $R^2(\cos^2\alpha + \sin^2\alpha) = 4^2 + 5^2$

$R^2 = 41$

$R = \sqrt{41}$

② ÷ ①: $\tan\alpha = 5/4$

$\alpha = 0.8961$ (1st quadrant)

$\therefore \sqrt{41}\sin(\theta + 0.8961) = 3$

$\sin(\theta + 0.8961) = \frac{3}{\sqrt{41}}$

$\theta + 0.8961 = 0.4876, 2.6540$

$\theta = -0.4084, 1.7579$

$\boxed{\theta = 1.758, 5.875}$ [3]

Otherwise: Let $t = \tan\frac{\theta}{2}$

$$\frac{8t}{1+t^2} + \frac{5(1-t^2)}{1+t^2} = 3$$

$8t + 5 - 5t^2 = 3 + 3t^2$

$8t^2 - 8t - 2 = 0$

$4t^2 - 4t - 1 = 0$

$(2t-1)^2 = 2$

$2t - 1 = \pm\sqrt{2}$

$2t = 1 \pm \sqrt{2}$

$\tan\frac{\theta}{2} = \frac{1 \pm \sqrt{2}}{2}$

$\frac{\theta}{2} = -0.2042, 0.8790$

$\theta = -0.4084, 1.7579$

$\boxed{\theta = 1.758, 5.875}$ [3]

Generally well done.

(i) Students are reminded that they need to SHOW their working for how they get $R=\sqrt{41}$ not just $R=\sqrt{4^2+5^2}$. A few students also left $R=\pm\sqrt{41}$.

(ii) Students also did not mention what quadrant α was in.

(iii) Many students lost marks for not considering the domain when solving the equation.

(iv) Many students still cannot work in radians and used degrees when the question clearly gave the domain in radians. Other students worked in degrees and then converted to radians at the last step which resulted in rounding errors (they were not penalised), however students need to practise working in radians rather than converting if the question is given to them in radians.

(b) (i) $V=V_0(1-e^{-kt}) \Rightarrow V_0 - V = V_0 e^{-kt}$

$\frac{dV}{dt} = V_0 (ke^{kt})$

$= k \cdot V_0 e^{kt}$

$= k(V_0 - V)$ [1]

Many students could not do this but hopefully this changes after doing differential equations.

(ii) $t=5,\ V=\frac{V_0}{4}$: $\frac{\cancel{V_0}}{4} = \cancel{V_0}(1-e^{-5k})$

$\frac{1}{4} = 1 - e^{-5k}$

$e^{-5k} = \frac{3}{4}$

$-5k = \ln\frac{3}{4}$

$k = -\frac{1}{5}\ln\frac{3}{4}$

$V=\frac{V_0}{2}$: $\frac{\cancel{V_0}}{2} = \cancel{V_0}(1-e^{\frac{1}{5}\ln\frac{3}{4}t})$

$\frac{1}{2} = 1 - \left(\frac{3}{4}\right)^{t/5}$

$\left(\frac{3}{4}\right)^{t/5} = \frac{1}{2}$

$\frac{t}{5} = \frac{\ln 1/2}{\ln 3/4}$

$t = 12.047$

$\therefore$ another 7.0 hours.

Done very well. Some students missed out the last part of the question and just found the time it took for half the water to evaporate.

(c) RTP: $\binom{2n}{3} = 2\binom{n}{3} + 2n\binom{n}{3}$

LHS = no. ways of choosing 3 students from $2n$ students $= \binom{2n}{3}$

RHS: 4 cases: Choose 3 boys from n: $\binom{n}{3}$ ways

or Choose 2 boys from n & 1 girl from n: $\binom{n}{2}\cdot\binom{n}{1} = n\binom{n}{2}$

Other 2 cases identical for more girls than boys

$\therefore$ Total ways $= 2\left[\binom{n}{3} + n\cdot\binom{n}{3}\right]$

$= 2\binom{n}{3} + 2n\binom{n}{2}$

As we are counting the same thing on LHS & RHS, result holds.

[3]

Most students were able to do this question or at the very least the LHS. Some students lost marks for the RHS because their explanation was not clear enough or they didn't show their calculations. As the expression was given in the question, the expression had to be very clearly explained or calculations had to be shown. The best responses just listed all the 4 cases with their calculations and added them together.

(d) (i) $V = \pi\int_{\alpha}^{\frac{3\pi}{2}} (1+\sin x)^2\,dx$

$= \pi\int_{\alpha}^{\frac{3\pi}{2}} (1 + 2\sin x + \sin^2 x)\,dx$

$= \pi\int_{\alpha}^{\frac{3\pi}{2}} \left(1 + 2\sin x + \tfrac{1}{2}[1-\cos 2x]\right)dx$

$= \pi\int_{\alpha}^{\frac{3\pi}{2}} \left(\tfrac{3}{2} + 2\sin x - \tfrac{1}{2}\cos 2x\right)dx$

$= \pi\left[\tfrac{3}{2}x - 2\cos x - \tfrac{1}{4}\sin 2x\right]_{\alpha}^{3\pi/2}$

$= \pi\left\{\left(\tfrac{9\pi}{4} - 0 - 0\right) - \left(\tfrac{3\alpha}{2} - 2\cos\alpha - \tfrac{1}{4}\sin 2\alpha\right)\right\}$

$= \frac{\pi}{4}\left(9\pi - 6\alpha + 8\cos\alpha + \sin 2\alpha\right)$ units^3

[3]

Students are reminded to read the question carefully to determine exactly what the question is asking them to find. Many students were able to write out the integral but gave the wrong bounds which meant they were not able to arrive the required answer. For the students who were able to write down the correct bounds, they are reminded that for SHOW questions, they need to SHOW their working i.e. not skip steps when substituting in the bounds and simplifying. Some students combined multiple steps when moving from one line to another. You should simplify the expression fully that is inside the brackets before factorising out further, not do both of these in the one line.

(ii) $\frac{3\pi}{2} - u = \frac{\pi}{2} \Rightarrow u = \pi$

$\frac{dV}{dt} = \frac{1}{2}$ (given)

$\frac{dV}{du} = \frac{\pi}{4}\left(-6 - 8\sin u + 2\cos 2u\right) \Rightarrow \left.\frac{dV}{du}\right|_{u=\pi} = \frac{\pi}{4}\left(-6 - 0 + 2\right) = -\pi$

$$\left.\frac{du}{dt}\right|_{u=\pi} = \left.\frac{du}{dV}\right|_{u=\pi} \times \left.\frac{dV}{dt}\right|_{u=\pi}$$

$$= -\frac{1}{\pi} \times \frac{1}{2}$$

$$= -\frac{1}{2\pi}$$

but height $h = \frac{3\pi}{2} - u$

$$\frac{dh}{dt} = -\frac{du}{dt}$$

$$= \frac{1}{2\pi} \text{ cm/s}$$

Done very poorly. Most students got 1 mark for this question. Majority of students substituted the wrong value into their equation. When the height is $\frac{\pi}{2}$, $u = \pi$. Also, many students fudged their answer to say that the height was increasing at a rate of … even though they got a negative answer which indicated decreasing. Very few students managed to justify this last part.